The transition from diffusion to blow-up for a nonlinear Schrödinger equation in dimension 1

2005 J. Phys. A: Math. Gen. 38 8379

(http://iopscience.iop.org/0305-4470/38/39/006)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 8379–8392 doi:10.1088/0305-4470/38/39/006

The transition from diffusion to blow-up for a nonlinear Schrödinger equation in dimension 1

Riccardo Adami[1] **and Andrea Sacchetti**[2]

[1] Dipartimento Matematica, Universitá di Roma 'La Sapienza', Piazzale Aldo Moro 2, 00185 Roma, Italy
[2] Dipartimento di Matematica Pura ed Applicata, Universitá di Modena e Reggio Emilia, Via Campi 213/B, Modena 41100, Italy

E-mail: Adami@mat.uniroma1.it and Sacchetti@unimore.it

Received 1 June 2005, in final form 28 July 2005
Published 14 September 2005
Online at stacks.iop.org/JPhysA/38/8379

Abstract
We consider the time-dependent one-dimensional nonlinear Schrödinger equation with a pointwise singular potential. We prove that if the strength of the nonlinear term is small enough, then the solution is well defined for any time, regardless of the choice of initial data; in contrast, if the nonlinearity power is larger than a critical value, for some initial data a blow-up phenomenon occurs in finite time. In particular, if the system is initially prepared in the ground state of the linear part of the Hamiltonian, then we obtain an explicit condition on the parameters for the occurrence of the blow-up.

PACS numbers: 03.65.Db, 03.75.−b, 02.30.Jr, 05.45.Yv
Mathematics Subject Classification: 35Q55, 35B40, 81Q05

1. Introduction

In this paper, we study the dynamics of a one-dimensional quantum particle subject to a linear point interaction and to a nonlinearity of the power type. The time evolution of the system is then ruled by the time-dependent nonlinear Schrödinger (NLS) equation

$$\begin{cases} \mathrm{i}\dot{\psi}_t = H\psi_t, & H\psi = -\psi'' + V\psi \\ \psi_t(x)|_{t=t_0} = \psi_0(x) \end{cases}, \qquad \dot{\psi}_t = \frac{\partial \psi_t}{\partial t} \quad \text{and} \quad \psi_t' = \frac{\partial \psi_t}{\partial x} \tag{1}$$

where ψ is a square integrable function on the real line and $V = V(|\psi|, x)$ is a nonlinear potential. In particular, we restrict ourselves to the case

$$V\psi = \gamma\psi\delta_0 + \rho|\psi|^{2\mu}\psi, \qquad \mu \geqslant 0, \quad \gamma, \quad \rho \in \mathbb{R}, \tag{2}$$

where δ_0 is the Dirac's delta pointwise interaction at $x = 0$.

First introduced to describe the propagation of laser beams (see, e.g., [18]), NLS equations have revealed themselves to be a quite flexible and far reaching tool. In particular, the one-dimensional NLS with cubic nonlinearity (namely $\mu = 1$) is currently used to model the dynamics of quasi one-dimensional Bose–Einstein condensates (see, e.g., [15, 13]) and the propagation of laser pulses in nonlinear media. In addition, it is worth mentioning that the case $\mu = 1/2$ is also significant, at least in experiments (see, e.g., [12]).

On the other hand, Dirac's delta potentials provide a general and idealized model for short-range interactions. Introduced by Fermi in three dimensions in order to investigate the scattering of slow neutrons by atoms ([10]), such potentials were later recognized to provide the simplest example of exactly solvable quantum models and have been widely employed in toy models.

The model (1)–(2) was introduced by Witthaut, Mossmann and Korsch [19] in order to describe at a phenomenological level the effect of short-range obstacles in models of nonlinear transport. The main topics of such investigation were the properties of the stationary states. Our contribution is complementary to their results, and focuses on well-posedness, conservation laws, and study of the blow-up phenomenon.

The spirit of this paper is the same of [1], where the authors introduced a one-dimensional model with nonlinear point interactions and investigated the related blow-up phenomenon, and of [2, 3], where the results were extended to a three-dimensional setting. On the other hand, it is worth recalling that the dynamics of a quantum particle under the action of a quadratic potential (attractive or repulsive) has been extensively studied in a series of works by Carles (see, e.g., [6, 7]) with peculiar care on the effects of the external potential to the occurrence of blow-up and to the blow-up time. In this respect our investigation permits us to say that, if the nonlinear term is attractive, the nonlinearity power is greater than 4, its strength is greater than or equal to a critical value, and the energy is negative, then a linear point interaction cannot prevent the blow-up.

The basic observation is that the time propagator of the free Laplacian with a delta interaction restricted to the space associated with its absolutely continuous spectrum satisfies some dispersive estimates and consequently some inequality of the Strichartz's type. Moreover, the energy space associated with a delta potential coincides with $H^1(\mathbb{R})$. Let us stress that this last feature fails in the three-dimensional setting, due to the emergence of a singularity of type $1/|x|$, and this prevents us from extending our model to three dimensions in a trivial way.

The paper is organized as follows. In section 2 we introduce the model, in section 3 we prove the well-posedness of the problem in the energy space and show the conservation laws of the L^2-norm and of the energy. Section 4 is devoted to the study of the blow-up phenomenon, and finally section 5 contains some concluding remarks.

We quoted some formal computations also, in order to exhibit the simple structure of some results that would risk to remain hidden by the technicalities of the rigorous proofs, and to show that to some extent the delta potential is well behaved, at least when supported on manifolds of codimension 1.

Concerning notation, we shall write $\|\cdot\|_p$ for the norm in the space $L^p(\mathbb{R})$. In the case $p = 2$ the index in the symbol of the norm will be omitted.

2. NLS equation with a pointwise interaction

Consider the problem defined in (1), (2). It is well known (see, e.g., [5]) that when the nonlinear term is absent, that is, $\rho = 0$, the operator

$$H_\gamma = -\psi'' + \gamma\psi\delta_0$$

is self-adjoint on the domain $H^2(\mathbb{R}\backslash\{0\})$ with boundary conditions

$$\psi(0+0) = \psi(0-0) \tag{3}$$

and

$$\psi'(x+0) - \psi'(x-0) = \gamma\psi(0+0). \tag{4}$$

Namely

$$D(H_\gamma) = \{\psi \in H^2(\mathbb{R}\backslash\{0\}) : \psi \text{ satisfies (3) and (4)}\}.$$

Remark 1. Note that, due to (3), the function $\psi \in D(H_\gamma)$ is continuous in $x = 0$ and therefore $D(H_\gamma)$ is a subspace of $H^1(\mathbb{R})$. Thus, in the following we denote by $\psi(0)$ the limit (3).

Let us recall some basic properties of the spectrum of H_γ. For details see [5, 4].

The essential spectrum of H_γ is purely absolutely continuous and coincides with the positive real axis:

$$\sigma_{\text{ess}}(H_\gamma) = \sigma_{\text{ac}}(H_\gamma) = [0, +\infty).$$

Moreover,

- If $\gamma \geqslant 0$ then the discrete spectrum of H_γ is empty;
- If $\gamma < 0$ then the discrete spectrum of H_γ is given by just one simple eigenvalue

$$\lambda = -\tfrac{1}{4}\gamma^2$$

with the associated normalized eigenvector

$$\phi_\gamma(x) = \sqrt{\frac{|\gamma|}{2}}\,\mathrm{e}^{-|\gamma|\cdot|x|/2}.$$

In addition, we shall make use of the explicit expression of the time evolution generated by H_γ, that is, an integral operator whose kernel reads [17, 4]

$$U_\gamma^t(x,y) = U_0^t(x-y) + \begin{cases} -\frac{\gamma}{2}\int_0^{+\infty} \mathrm{d}u\, \mathrm{e}^{-\frac{\gamma}{2}u} U_0^t(u+|x|+|y|), & \gamma > 0 \\ 0, & \gamma = 0 \\ \mathrm{e}^{\mathrm{i}\frac{\gamma^2}{4}t}\phi_\gamma(x)\phi_\gamma(y) + \frac{\gamma}{2}\int_0^{+\infty} \mathrm{d}u\, \mathrm{e}^{\frac{\gamma}{2}u} U_0^t(u-|x|-|y|) & \gamma < 0 \end{cases} \tag{5}$$

where U_0^t is the integral kernel associated with the free Laplacian, namely

$$U_0^t(\zeta) = \frac{1}{\sqrt{4\pi\mathrm{i}t}} \exp\left(-\frac{|\zeta|^2}{4\mathrm{i}t}\right).$$

3. Local existence and conservation laws

Theorem 1. *If the initial data ψ_0 belong to $H^1(\mathbb{R})$ then the Cauchy problem (1) with potential (2) admits a unique local solution*

$$\psi \in \mathcal{H} = C((-T_{\min}, T_{\max}), H^1(\mathbb{R})) \cap C^1((-T_{\min}, T_{\max}), H^{-1}(\mathbb{R}))$$

for some $T_{\min}, T_{\max} > 0$, satisfying the boundary conditions (3) and (4) for almost all t.

Furthermore, the following conservation laws hold:

- *Conservation of the norm*

$$\mathcal{N}[\psi_t] = \mathcal{N}[\psi_0] = \|\psi_0\|.$$

- *Conservation of the energy*

$$\mathcal{E}[\psi_t] = \mathcal{E}_{\rm lin}[\psi_t] + \frac{\rho}{\mu+1}\|\psi_t\|_{2\mu+2}^{2\mu+2}, \qquad \mathcal{E}_{\rm lin}[\psi] = \|\psi'\|^2 + \gamma|\psi(0)|^2. \tag{6}$$

Proof. We split the proof of the theorem in three steps.

Strichartz-type estimate. As a first step, we remark that the Cauchy problem (1) is equivalent to the Duhamel formula

$$\psi_t = \mathrm{e}^{-\mathrm{i}tH_\gamma}\psi_0 - i\rho\int_0^t \mathrm{e}^{-\mathrm{i}(t-s)H_\gamma}|\psi_s|^{2\mu}\psi_s\,\mathrm{d}s \tag{7}$$

and that the evolution operator can be written as

$$\mathrm{e}^{-\mathrm{i}H_\gamma t}\psi = \begin{cases} \mathrm{e}^{\mathrm{i}\gamma^2 t/4}\langle\psi,\phi_\gamma\rangle\phi_\gamma + \mathrm{e}^{-\mathrm{i}H_\gamma t}P_c\psi & \text{if} \quad \gamma < 0 \\ \mathrm{e}^{-\mathrm{i}H_\gamma t}P_c\psi & \text{if} \quad \gamma \geqslant 0 \end{cases}$$

where P_c is the orthogonal projection in $L^2(\mathbb{R})$ onto the subspace orthogonal to ϕ_γ.

We are now ready to prove the basic dispersive estimate.

Lemma 1. *For any $\varphi \in L^1(\mathbb{R})$ the following estimate holds:*

$$\|\mathrm{e}^{-\mathrm{i}tH_\gamma}P_c\varphi\|_\infty \leqslant (\pi t)^{-1/2}\|\varphi\|_1. \tag{8}$$

Proof. From expression (5) of the time evolution generated by H_γ we have that the operator $\mathrm{e}^{-\mathrm{i}H_\gamma t}P_c$ is represented by the integral kernel

$$[\mathrm{e}^{-\mathrm{i}H_\gamma t}P_c](x,y) = U_0^t(x-y) - \frac{|\gamma|}{2}\int_0^{+\infty}\mathrm{d}u\,\mathrm{e}^{-\frac{|\gamma|}{2}u}U_0^t(u\pm|x|\pm|y|)$$

where the '+' signs hold if $\gamma > 0$ and the '−' signs hold if $\gamma < 0$.

Then, given $\varphi \in L^1(\mathbb{R})$ we conclude that

$$\begin{aligned} |[\mathrm{e}^{-\mathrm{i}H_\gamma t}P_c\varphi](x)| &\leqslant \left|\int_{\mathbb{R}}\frac{\exp\left(\frac{|x-y|^2}{4\mathrm{i}t}\right)}{\sqrt{4\pi\mathrm{i}t}}\varphi(y)\,\mathrm{d}y\right| \\ &\quad + \frac{|\gamma|}{2}\left|\int_{\mathbb{R}}\mathrm{d}y\int_0^{+\infty}\mathrm{d}u\,\mathrm{e}^{-\frac{|\gamma|}{2}u}\frac{\exp\left(\frac{(u\pm|x|\pm|y|)^2}{4\mathrm{i}t}\right)}{\sqrt{4\pi\mathrm{i}t}}\varphi(y)\right| \\ &\leqslant \frac{1}{\sqrt{4\pi t}}\|\varphi\|_1 + \frac{|\gamma|}{4\sqrt{\pi t}}\int_{\mathbb{R}}|\varphi(y)|\mathrm{d}y\int_0^{+\infty}\mathrm{e}^{-\frac{|\gamma|}{2}u}\mathrm{d}u \\ &\leqslant \frac{1}{\sqrt{\pi t}}\|\varphi\|_1. \end{aligned}$$

□

Remark 2. Estimate (8) together with the obvious inequality

$$\|\mathrm{e}^{-\mathrm{i}tH_\gamma}P_c\varphi\| \leqslant \|\varphi\|$$

and the Riesz–Thorin interpolation theorem, yields

$$\|\mathrm{e}^{-\mathrm{i}H_\gamma t}P_c\varphi\|_p \leqslant (\pi t)^{-d\left(\frac{1}{2}-\frac{1}{p}\right)}\|\varphi\|_{p'} \tag{9}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $2 \leqslant p \leqslant \infty$. Moreover, (9) gives the following Strichartz-type inequality: for any pair (q, r) with $r \geqslant 2$ and $q = \frac{4r}{r-2}$, $\varphi \in L^2(\mathbb{R})$ and $T > 0$ there exists a positive constant C such that

$$\|\mathrm{e}^{-\mathrm{i}H_\gamma t}P_c\varphi\|_{L^q((-T,T),L^r(\mathbb{R}))} \leqslant C\|\varphi\|. \tag{10}$$

Indeed, from the spectral resolution outlined in section 2, we know that H_γ is a self-adjoint operator bounded from below (see, e.g., [5, 16] for details). Then theorem 2.7.1 in [8] applies and yields (10).

Local existence and uniqueness. Let us define

$$K_\gamma = \begin{cases} H_\gamma & \text{if} \quad \gamma \geqslant 0 \\ H_\gamma + \dfrac{\gamma^2}{4} & \text{if} \quad \gamma < 0. \end{cases}$$

Clearly, $K_\gamma \geqslant 0$.

We prove the local existence and uniqueness for the solution to the equation

$$\psi_t = \mathrm{e}^{-\mathrm{i}tK_\gamma}\psi_0 - \mathrm{i}\rho\int_0^t \mathrm{e}^{-\mathrm{i}(t-s)K_\gamma}|\psi_s|^{2\mu}\psi_s\,\mathrm{d}s$$

from which local existence and uniqueness for the solution to equation (7) immediately follow. Uniqueness is guaranteed by the Strichartz-type estimate (10), as one can check following, for instance, the proof of proposition 4.2.1 in [8]. On the other hand, in order to prove existence we apply theorem 3.7.1 in [8] (see also sections 3.7 and 4.12 in the same reference), and therefore we just have to prove the following lemma.

Lemma 2. *The self-adjoint operator K_γ satisfies the following conditions:*

(i) $X_K \hookrightarrow L^p$, for any $p \in [2, +\infty]$, where X_K is the closure of the domain $D(K_\gamma) = D(H_\gamma)$ with respect to the norm induced by

$$\|u\|_{K_\gamma}^2 = \begin{cases} \|u'\|^2 + \|u\|^2 + \gamma|u(0)|^2 & \textit{if} \quad \gamma \geqslant 0 \\ \|u'\|^2 + \left(1 + \dfrac{\gamma^2}{4}\right)\|u\|^2 + \gamma|u(0)|^2 & \textit{if} \quad \gamma < 0. \end{cases}$$

(ii) The resolvent

$$[1 - \epsilon K_\gamma]^{-1} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$$

is continuous for any $\epsilon < 0$ and any $p \in [2, +\infty]$ and fulfils

$$\|[1 - \epsilon K_\gamma]^{-1}u\|_p \leqslant C\|u\|_p \tag{11}$$

for some C independent of $\epsilon < 0$.

Proof. Condition (i) immediately follows from estimate (A.1)

$$|u(0)|^2 \leqslant \|u\|_\infty^2 \leqslant \|u'\| \cdot \|u\|, \qquad \forall\, u \in D(K_\gamma).$$

Indeed, such estimate implies the equivalence of the norm $\|\cdot\|_{K_\gamma}$ and the ordinary norm in $H^1(\mathbb{R})$; hence, $X_K = H^1(\mathbb{R}) \hookrightarrow L^p(\mathbb{R})$ for any $p \in [2, +\infty]$. Condition (ii) follows by a direct computation, making use of the fact that the resolvent operator $[H_\gamma - \zeta]^{-1}$ is an integral operator

$$([H_\gamma - \zeta]^{-1}\phi)(x) = \int_{\mathbb{R}} \mathcal{G}(x, y; k)\phi(y)\,\mathrm{d}y, \qquad \zeta = k^2, \quad \operatorname{Im} k \geqslant 0,$$

with kernel (see, e.g., [5])

$$\mathcal{G}(x, y; k) = \frac{\mathrm{i}}{2k}\,\mathrm{e}^{\mathrm{i}k|x-y|} + \frac{1}{2k}\frac{\gamma}{i\gamma + 2k}\,\mathrm{e}^{\mathrm{i}k(|x|+|y|)}.$$

To prove (11) we define

$$\zeta = \begin{cases} \epsilon^{-1} & \text{if} \quad \gamma \geqslant 0 \\ \epsilon^{-1} - \dfrac{\gamma^2}{4} & \text{if} \quad \gamma < 0 \end{cases}, \qquad \zeta < 0,$$

and

$$k = \sqrt{\zeta} = \mathrm{i}\lambda, \qquad \lambda > \begin{cases} 0 & \text{if} \quad \gamma \geqslant 0 \\ -\dfrac{\gamma}{2} & \text{if} \quad \gamma < 0 \end{cases}.$$

Then

$$([1 - \epsilon K_\gamma]^{-1}\phi)(x) = -\epsilon^{-1}([H_\gamma - \zeta]^{-1}\phi)(x) = -\epsilon^{-1}\int_{\mathbb{R}} \frac{\mathrm{i}}{2k}\,\mathrm{e}^{\mathrm{i}k|x-y|}\phi(y)\,\mathrm{d}y + \\ -\epsilon^{-1}\int_{\mathbb{R}} \frac{1}{2k}\frac{\gamma}{\mathrm{i}\gamma + 2k}\,\mathrm{e}^{\mathrm{i}k(|x|+|y|)}\phi(y)\,\mathrm{d}y = I + II$$

where

$$I = -\frac{1}{2\lambda\epsilon}\int_{\mathbb{R}} \mathrm{e}^{-\lambda|x-y|}\phi(y)\,\mathrm{d}y$$
$$II = \frac{1}{2\lambda\epsilon}\frac{\mathrm{e}^{-\lambda|x|}\gamma}{\gamma + 2\lambda}\int_{\mathbb{R}} \mathrm{e}^{-\lambda|y|}\phi(y)\,\mathrm{d}y$$

Thus

$$\begin{aligned}
\|II\|_p &= \frac{1}{2\lambda|\epsilon|}\frac{\|\mathrm{e}^{-\lambda|\cdot|}\|_p|\gamma|}{|\gamma + 2\lambda|}\left|\int_{\mathbb{R}} \mathrm{e}^{-\lambda|y|}\phi(y)\,\mathrm{d}y\right| \\
&\leqslant \frac{2^{1/p}}{2\lambda^{(1+p)/p}p^{1/p}|\epsilon|}\frac{|\gamma|}{|\gamma + 2\lambda|}\|\mathrm{e}^{-\lambda|\cdot|}\phi\|_1 \\
&\leqslant \frac{2^{1/p}}{2\lambda^{(1+p)/p}p^{1/p}|\epsilon|}\frac{|\gamma|}{|\gamma + 2\lambda|}\|\mathrm{e}^{-\lambda|\cdot|}\|_{p'}\|\phi\|_p \\
&\leqslant \frac{2^{1/p}}{2\lambda^{(1+p)/p}p^{1/p}|\epsilon|}\frac{|\gamma|}{|\gamma + 2\lambda|}\left[\frac{2}{\lambda p'}\right]^{1/p'}\|\phi\|_p \\
&\leqslant \frac{|\gamma|}{p^{1/p}p'^{1/p'}}\frac{1}{\lambda^2|\epsilon||\gamma + 2\lambda|}\|\phi\|_p
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and

$$\lambda^2|\epsilon||\gamma + 2\lambda| = \begin{cases} |\gamma + 2\lambda| > \gamma & \text{if} \quad \gamma \geqslant 0 \\ \dfrac{4 + |\epsilon|\gamma^2}{|\gamma|\sqrt{|\epsilon|}[\sqrt{4 + |\epsilon|\gamma^2} + |\gamma|\sqrt{|\epsilon|}]}\dfrac{|\gamma|}{4} \geqslant \dfrac{|\gamma|}{8} & \text{if} \quad \gamma < 0 \end{cases},$$

Hence

$$\|II\|_p \leqslant C_{II}\|\phi\|_p, \qquad C_{II} = \frac{8}{p^{1/p}p'^{1/p'}}$$

The integral term I, corresponding to the free Laplacian, is similarly treated,

$$\|I\|_p \leqslant \frac{1}{2\lambda|\epsilon|}\|\mathrm{e}^{-\lambda|\cdot|} \star \phi\|_p \leqslant \frac{1}{2\lambda|\epsilon|}\|\mathrm{e}^{-\lambda|\cdot|}\|_1\|\phi\|_p \leqslant \frac{1}{\lambda^2|\epsilon|}\|\phi\|_p$$

where $\lambda^2|\epsilon| \geqslant 1$ for any ϵ. □

Conservation laws. Following 3.7.1 in [8], conservation of the L^2-norm and of the energy (6) follow from points (i) and (ii), so the proof is complete. □

Remark 3. Our result on conservation laws can be *formally* extended to the case of a *nonlinear* pointwise interaction (see, e.g., [1]) with potential

$$V_\sigma\psi = \gamma|\psi|^{2\sigma}\psi\delta_0 + \rho|\psi|^{2\mu}\psi.$$

In such a case the energy is defined as

$$\mathcal{E}_\sigma[\psi] = \|\psi'\|^2 + \frac{\rho}{\mu+1}\|\psi\|_{2\mu+2}^{2\mu+2} + \frac{\gamma}{\sigma+1}|\psi(0)|^{2\sigma+2}. \tag{12}$$

Since a formal proof of conservation of the norm is fairly easy, we deal with conservation of the energy only. Let ψ be regular enough, then (for the sake of simplicity let us denote ψ_t by ψ)

$$\begin{aligned}\frac{\mathrm{d}\|\psi'\|^2}{\mathrm{d}t} &= 2\,\mathrm{Re}\int_{\mathbb{R}} \bar{\psi}'(\dot{\psi})'\,\mathrm{d}x = 2\,\mathrm{Re}\left[\mathrm{i}\int_{\mathbb{R}} \bar{\psi}''(\mathrm{i}\dot{\psi})\,\mathrm{d}x\right]\\ &= 2\,\mathrm{Re}\left[\mathrm{i}\int_{\mathbb{R}} \bar{\psi}''[-\psi'' + \rho|\psi|^{2\mu}\psi + \gamma|\psi|^{2\sigma}\delta_0\psi]\,\mathrm{d}x\right]\\ &= -2\rho\,\mathrm{Im}\left[\int_{\mathbb{R}} \bar{\psi}''|\psi|^{2\mu}\psi\,\mathrm{d}x\right] - 2\gamma\,\mathrm{Im}\left[\int_{\mathbb{R}} \bar{\psi}''\psi|\psi|^{2\sigma}\delta_0\,\mathrm{d}x\right]\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\rho}{\mu+1}\|\psi\|_{2\mu+2}^{2\mu+2}\right) &= 2\frac{\rho}{\mu+1}\,\mathrm{Re}\int_{\mathbb{R}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\psi^{\mu+1}\right)\bar{\psi}^{\mu+1}\,\mathrm{d}x\\ &= 2\rho\,\mathrm{Re}\left[-\mathrm{i}\int_{\mathbb{R}} \psi^{\mu}\bar{\psi}^{\mu+1}(\mathrm{i}\dot{\psi})\,\mathrm{d}x\right]\\ &= 2\rho\,\mathrm{Im}\left[\int_{\mathbb{R}} |\psi|^{2\mu}(-\bar{\psi}\psi'' + \rho|\psi|^{2\mu+2} + \gamma|\psi|^{2\sigma+2}\delta_0)\,\mathrm{d}x\right]\\ &= -2\rho\,\mathrm{Im}\int_{\mathbb{R}} |\psi|^{2\mu}\bar{\psi}\psi''\,\mathrm{d}x = 2\rho\,\mathrm{Im}\int_{\mathbb{R}} |\psi|^{2\mu}\psi\bar{\psi}''\,\mathrm{d}x\end{aligned}$$

and similarly

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\gamma}{\sigma+1}|\psi(0)|^{2\sigma+2}\right) &= \frac{\gamma}{\sigma+1}\int_{\mathbb{R}}\delta_0 2\,\mathrm{Re}\left[\bar{\psi}^{\sigma+1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\psi^{\sigma+1}\right)\mathrm{d}x\right]\\ &= 2\gamma\,\mathrm{Im}\left[\int_{\mathbb{R}}\delta_0|\psi|^{2\sigma}\bar{\psi}''\psi\,\mathrm{d}x\right]\end{aligned}$$

Collecting all these facts we finally obtain that $\frac{d\mathcal{E}_\sigma}{dt} = 0$. In particular, for $\sigma = 0$, we obtain again the conservation of the energy (6).

Remark 4. As in the linear case, the continuity equation is

$$\partial_t|\psi_t|^2 + \partial_x[2\,\mathrm{Im}\,\overline{\psi_t}\partial_x\psi_t] = 0. \tag{13}$$

The proof is elementary. From (13) one can give another proof of the conservation of the L^2-norm.

4. Analysis of the blow-up

The phenomenon of the blow-up has been investigated extensively in the case of the standard nonlinear Schrödinger equation.The most advanced results on this topic are due to Merle and Raphael ([14]).

In our problem the definition of blow-up is the same as in the standard case.

Definition 1. *Let $\psi \in C((-T_{\min}, T_{\max}), H^1(\mathbb{R})) \cap C^1((-T_{\min}, T_{\max}), H^{-1}(\mathbb{R}))$ be the unique maximal solution of the Cauchy problem (1) with a nonlinear potential V like in (2) and initial* *data $\psi_0 \in H^1(\mathbb{R})$. We call ψ a blow-up solution and say that ψ blows up in finite time if there exists a finite $T_\star$ such that*

$$\lim_{t\to T_\star} \|\psi_t\|_{H^1} = +\infty.$$

If $T_\star > 0$ then we say that ψ blows up forwards in time, if $T_\star < 0$ then we say that it blows up backwards in time.

Remark 5. As in the standard case, we have a blow-up alternative, namely, either the solution is global in time, or it blows up in finite time. This can be proven following theorem 3.3.9, step 2 in [8]. As a consequence, if ψ blows up forwards (backwards) in time, then $T_\star = T_{\max}(T_{\min})$.

We follow the method of Glassey [11], based on the computation of the moment of inertia of the solution. To this end, let us define the set

$$\mathcal{K} = H^1(\mathbb{R}) \cap \{\psi \in L^2(\mathbb{R}) : x\psi \in L^2(\mathbb{R})\} \cap \{\psi \in L^2(\mathbb{R}) : \|\psi\| = 1\}.$$

The result we prove reads as follows.

Theorem 2. *Let $\psi_0 \in H^1(\mathbb{R})$ be the initial data for the Cauchy problem (1) with a nonlinear potential V like in (2). Let $\psi \in (-T_{\min}, T_{\max})$ be the solution to such problem. We define the variance $I(t)$ (also called moment of inertia) of the solution ψ as follows:*

$$I = I(t) = \int_{\mathbb{R}} x^2 |\psi_t(x)|^2 \,\mathrm{d}x, \qquad t \in (-T_{\min}, T_{\max}).$$

Then, $I \in C^2(-T_{\min}, T_{\max})$ and

$$\dot{I} = 4\,\mathrm{Im} \int_{\mathbb{R}} x\psi_t'(x)\bar{\psi}_t(x)\,\mathrm{d}x, \qquad \psi \in \mathcal{H} \tag{14}$$

$$\ddot{I} = 8\mathcal{E}[\psi_t] + 4\rho\frac{\mu-2}{\mu+1}\|\psi_t\|_{2\mu+2}^{2\mu+2} - 4\gamma|\psi_t(0)|^2. \tag{15}$$

Proof. In order to prove that $x\psi_t \in L^2(\mathbb{R})$ and equality (14), standard regularization arguments apply. In particular, we set

$$I^\epsilon(t) = I_-^\epsilon(t) + I_+^\epsilon(t)$$

where $\epsilon > 0$, $\mathbb{R}_+ = (0, +\infty)$, $\mathbb{R}_- = (-\infty, 0)$, $\theta_\epsilon(x) = \mathrm{e}^{-\epsilon x^2}$ and

$$I_-^\epsilon(t) = \int_{\mathbb{R}_-} x^2\theta_\epsilon(x)|\psi_t(x)|^2\,\mathrm{d}x, \qquad I_+(t) = \int_{\mathbb{R}_+} \theta_\epsilon(x)x^2|\psi_t(x)|^2\,\mathrm{d}x.$$

Hereafter, for the sake of simplicity, let us drop out the explicit dependence on t where this does not cause misunderstanding. By standard computations one finds that

$$\begin{aligned}
\dot{I}_+^\epsilon &= 2\,\mathrm{Re}\int_{\mathbb{R}_+}\theta_\epsilon(x)x^2\dot{\psi}\bar{\psi}\,\mathrm{d}x = 2\,\mathrm{Im}\int_{\mathbb{R}_+}\theta_\epsilon(x)x^2[-\psi'' + \rho|\psi|^{2\mu}\psi]\bar{\psi}\,\mathrm{d}x\\
&= -2\,\mathrm{Im}\int_{\mathbb{R}_+} x^2\theta_\epsilon(x)\psi''\bar{\psi}\,\mathrm{d}x\\
&= 2\,\mathrm{Im}\int_{\mathbb{R}_+}\theta_\epsilon(x)x^2\psi'\bar{\psi}'\,\mathrm{d}x + 4\,\mathrm{Im}\int_{\mathbb{R}_+}\theta_\epsilon(x)x\psi'\bar{\psi}\,\mathrm{d}x + 2\,\mathrm{Im}\int_{\mathbb{R}_+}2\epsilon x^3\theta_\epsilon(x)\psi'\bar{\psi}\,\mathrm{d}x\\
&= 4\,\mathrm{Im}\int_{\mathbb{R}_+}\theta_\epsilon(x)x\psi'\bar{\psi}\,\mathrm{d}x - 4\,\mathrm{Im}\int_{\mathbb{R}_+}\epsilon x^3\theta_\epsilon(x)\psi'\bar{\psi}\,\mathrm{d}x.
\end{aligned}$$

Since a similar result holds for $\dot{I}^{\epsilon}_{-}$, we can conclude that

$$\dot{I}^{\epsilon} = 4J^{\epsilon} - 4\,\mathrm{Im}\int_{\mathbb{R}} \epsilon x^3 \theta_\epsilon(x)\psi'\bar{\psi}\,\mathrm{d}x, \qquad J^{\epsilon} = \mathrm{Im}\int_{\mathbb{R}} \theta_\epsilon(x) x\psi'\bar{\psi}\,\mathrm{d}x.$$

Therefore, for any fixed t the variance I_ϵ is bounded as a function of $\epsilon > 0$ (for details see the proof of lemma 6.5.2 in [8]). Hence $x\psi_t \in L^2(\mathbb{R})$ and the limit $\epsilon \to 0$ gives (14). Now, in order to prove that $I \in C^2$ and the validity of formula (15) let us write again

$$J^{\epsilon} = J^{\epsilon}_{+} + J^{\epsilon}_{-}$$

where, for any fixed t,

$$\begin{aligned} \dot{J}^{\epsilon}_{\pm} &= -\mathrm{Im}\int_{\mathbb{R}_\pm} \dot{\psi}[2\theta_\epsilon x\bar{\psi}' + (\theta_\epsilon + x\theta_\epsilon')\bar{\psi}]\,\mathrm{d}x = F^{\epsilon}_{\pm}(\psi'', \psi', \psi) \\ F^{\epsilon}_{\pm}(\psi'', \psi', \psi) &= -\mathrm{Re}\int_{\mathbb{R}_\pm} (\psi'' - \rho|\psi|^{2\mu}\psi)[2\theta_\epsilon x\bar{\psi}' + (\theta_\epsilon + x\theta_\epsilon')\bar{\psi}]\,\mathrm{d}x. \end{aligned} \tag{16}$$

We have defined $F_\pm$ as functionals on $D(H_\gamma)$, but they can be extended to the whole space $H^1(\mathbb{R})$. Indeed, for any function $u \in H^2(\mathbb{R})$ satisfying the boundary conditions (3) and (4), an integration by parts gives

$$F^{\epsilon}_{+}(u'', u', u) + F^{\epsilon}_{-}(u'', u', u) = G^{\epsilon}(u', u) + \gamma|u(0)|^2 \tag{17}$$

where the right-hand side is well defined for any $u \in H^1(\mathbb{R})$. To see that, let us further integrate by parts and obtain

$$\int_{\mathbb{R}_+} \psi''\theta_\epsilon x\bar{\psi}'\,\mathrm{d}x = -\int_{\mathbb{R}_+} \psi'\theta_\epsilon x\bar{\psi}''\,\mathrm{d}x - \int_{\mathbb{R}_+} |\psi'|^2(\theta_\epsilon x)'\,\mathrm{d}x.$$

Therefore,

$$\mathrm{Re}\left[\int_{\mathbb{R}_+} \psi'' 2\theta_\epsilon x\bar{\psi}'\,\mathrm{d}x\right] = \frac{1}{2}\mathrm{Re}\left[-\int_{\mathbb{R}_+} |\psi'|^2(2\theta_\epsilon x)'\,\mathrm{d}x\right]. \tag{18}$$

Furthermore,

$$\int_{\mathbb{R}_+} \psi''(\theta_\epsilon + x\theta_\epsilon')\bar{\psi}\,\mathrm{d}x = -\psi'(0+0)\bar{\psi}(0+0) - \int_{\mathbb{R}_+} \psi'[(\theta_\epsilon + x\theta_\epsilon')\bar{\psi}]'\,\mathrm{d}x. \tag{19}$$

Collecting (16), (18) and (19) we can conclude that

$$\begin{aligned} F^{\epsilon}_{\pm} &= \pm\mathrm{Re}[\psi'(0\pm 0)\bar{\psi}(0\pm 0)] + G^{\epsilon}_{\pm} \\ G^{\epsilon}_{\pm} &= \frac{1}{2}\int_{\mathbb{R}_\pm} |\psi'|^2(2\theta_\epsilon x)'\,\mathrm{d}x + \mathrm{Re}\left[\int_{\mathbb{R}_\pm} \psi'[(\theta_\epsilon + x\theta_\epsilon')\bar{\psi}]'\,\mathrm{d}x\right] \\ &\qquad + \mathrm{Re}\int_{\mathbb{R}_\pm} \rho|\psi|^{2\mu}\psi[2\theta_\epsilon x\bar{\psi}' + (\theta_\epsilon + x\theta_\epsilon')\bar{\psi}]\,\mathrm{d}x \end{aligned}$$

and formula (17) follows since $G^{\epsilon} = G^{\epsilon}_{+} + G^{\epsilon}_{-}$. Thus, by means of a continuity argument we obtain that

$$\dot{J}^{\epsilon} = G^{\epsilon}(\psi_t', \psi_t) + \gamma|\psi_t(0)|^2$$

is well defined since $\psi_t \in H^1(\mathbb{R})$ for any t. Applying the dominate convergence theorem we compute the limit $\epsilon \to 0$ and conclude the proof.

Such a technical procedure justifies the following formal computation, too,

$$\begin{aligned}
\ddot I &= 4\,\mathrm{Im}\int_{\mathbb{R}}[x\psi'\dot{\bar\psi} + x\dot\psi'\bar\psi]\,\mathrm{d}x = 4\,\mathrm{Im}\int_{\mathbb{R}}[x\psi'\dot{\bar\psi} - \dot\psi\bar\psi - x\dot\psi\bar\psi']\,\mathrm{d}x\\
&= 8\,\mathrm{Im}\int_{\mathbb{R}} x\psi'\dot{\bar\psi}\,\mathrm{d}x - 4\,\mathrm{Im}\left(-\mathrm{i}\int_{\mathbb{R}}\mathrm{i}\dot\psi\bar\psi\,\mathrm{d}x\right)\\
&= -8\,\mathrm{Re}\int_{\mathbb{R}} x\psi'\bar\psi''\,\mathrm{d}x + 8\,\mathrm{Re}\int_{\mathbb{R}} x\psi'\bar V\bar\psi\,\mathrm{d}x + 4\,\mathrm{Re}\int_{\mathbb{R}}(H\psi)\bar\psi\,\mathrm{d}x\\
&= 4\,\mathrm{Re}\int_{\mathbb{R}}|\psi'|^2\mathrm{d}x + 4\,\mathrm{Re}\int_{\mathbb{R}} x(\psi''\bar\psi' - \psi'\bar\psi'')\,\mathrm{d}x + 8\,\mathrm{Re}\int_{\mathbb{R}} x\psi'\bar V\bar\psi\,\mathrm{d}x + 4\,\mathrm{Re}\int_{\mathbb{R}}(H\psi)\bar\psi\,\mathrm{d}x\\
&= 4\int_{\mathbb{R}}|\psi'|^2\mathrm{d}x + 8\,\mathrm{Re}\int_{\mathbb{R}} x\psi'\bar V\bar\psi\,\mathrm{d}x + 4\,\mathrm{Re}\int_{\mathbb{R}}(H\psi)\bar\psi\,\mathrm{d}x\\
&= 8\int_{\mathbb{R}}|\psi'|^2\mathrm{d}x + 4\int_{\mathbb{R}}\bar\psi V\psi\,\mathrm{d}x + 8\,\mathrm{Re}\int_{\mathbb{R}} x\psi'\bar V\bar\psi\,\mathrm{d}x
\end{aligned}$$

where $H\psi = -\psi'' + V\psi$, the potential V being given by (2). Hence

$$\begin{aligned}
\ddot I &= 8\int_{\mathbb{R}}|\psi'|^2\mathrm{d}x + 8\rho\,\mathrm{Re}\int_{\mathbb{R}} x\psi'\bar\psi|\psi|^{2\mu}\,\mathrm{d}x + 4\rho\int_{\mathbb{R}}|\psi|^{2\mu+2}\,\mathrm{d}x + 4\gamma\int_{\mathbb{R}}|\psi|^2\delta_0\,\mathrm{d}x\\
&= 8\left[\mathcal{E}[\psi] - \frac{\rho}{\mu+1}\|\psi\|_{2\mu+2}^{2\mu+2} - \gamma|\psi(0)|^2\right]\\
&\quad + 8\rho\,\mathrm{Re}\int_{\mathbb{R}} x\psi'\bar\psi|\psi|^{2\mu}\,\mathrm{d}x + 4\rho\|\psi\|_{2\mu+2}^{2\mu+2} + 4\gamma|\psi(0)|^2\\
&= 8\mathcal{E}[\psi] + 4\rho\frac{\mu-1}{\mu+1}\|\psi\|_{2\mu+2}^{2\mu+2} - 4\gamma|\psi(0)|^2 - \frac{4\rho}{\mu+1}\|\psi\|_{2\mu+2}^{2\mu+2}\\
&= 8\mathcal{E}[\psi] + 4\rho\frac{\mu-2}{\mu+1}\|\psi\|_{2\mu+2}^{2\mu+2} - 4\gamma|\psi(0)|^2
\end{aligned}$$

since

$$2(\mu+1)\,\mathrm{Re}\int_{\mathbb{R}} x\psi'\bar\psi|\psi|^{2\mu}\,\mathrm{d}x = -\|\psi\|_{2\mu+2}^{2\mu+2}.$$

□

Now, we are ready to carry out the analysis of the blow-up for the NLS equation with potential (2).

Theorem 3. *Let ψ_t be the solution of the Cauchy problem (1), (2), with initial data $\psi_0 \in \mathcal{K}$. If one of the following conditions,*

(i) $\rho \geqslant 0$,
(ii) $\rho < 0$ *and* $\mu < 2$,
(iii) $-\frac{3}{\tilde C} < \rho < 0$ *and* $\mu = 2$, *where* $\tilde C$ *is the positive constant appearing in estimate (A.2),*
(iv) $\rho_1 < \rho < 0$, *for some* $\rho_1 < 0$ *depending on* ψ_0, *and* $\mu > 2$, *is satisfied, then there is no blow-up and the solution* ψ *exists globally in time.*
In contrast, if
(v) $\mathcal{E}[\psi_0] < 0$, $\rho < \rho_2$ *for some* $\rho_2 < 0$, *and* $\mu > 2$,

then the solution ψ *blows up in finite time.*

Proof. Let C denote any positive constant and, for the sake of simplicity, let us drop in notation the explicit dependence on t (that is, let us write ψ instead of ψ_t, I instead of $I(t)$, and so on). From the conservation laws of the energy and of the norm and from inequality (A.1) it follows that

$$\begin{aligned}\mathcal{E}[\psi_0] &= \|\psi'\|^2 + \gamma|\psi(0)|^2 + \frac{\rho}{\mu+1}\|\psi\|_{2\mu+2}^{2\mu+2}\\ &\geqslant \|\psi'\|^2 - |\gamma||\psi(0)|^2 + \frac{\rho}{\mu+1}\|\psi\|_{2\mu+2}^{2\mu+2}\\ &\geqslant \|\psi'\|^2 - |\gamma|\|\psi\|_\infty^2 + \frac{\rho}{\mu+1}\|\psi\|_{2\mu+2}^{2\mu+2}\\ &\geqslant \|\psi'\|^2 - |\gamma|\|\psi'\| + \frac{\rho}{\mu+1}\|\psi\|_{2\mu+2}^{2\mu+2}\\ &\geqslant \left(\|\psi'\| - \frac{1}{2}|\gamma|\right)^2 - \frac{1}{4}\gamma^2 + \frac{\rho}{\mu+1}\|\psi\|_{2\mu+2}^{2\mu+2}\end{aligned} \tag{20}$$

from which, if $\rho \geqslant 0$, it immediately follows that

$$\|\psi'\| \leqslant C, \qquad \forall\, t \geqslant 0,$$

and then (i) is proved. On the other hand, if $\rho < 0$ then inequality (20) takes the form

$$\begin{aligned}\mathcal{E}[\psi_0] &\geqslant \|\psi'\|^2 - |\gamma|\|\psi\|_\infty^2 - \frac{|\rho|}{\mu+1}\|\psi\|_{2\mu+2}^{2\mu+2}\\ &\geqslant \|\psi'\|^2 - |\gamma|\|\psi'\| - \tilde{C}\frac{|\rho|}{\mu+1}\|\psi'\|^{\mu}\end{aligned} \tag{21}$$

where we make use of inequality (A.2). Therefore $\|\psi'\| \leqslant C, \forall\, t \geqslant 0$, provided that $\mu < 2$, so (ii) is proven. For $\mu = 2$ the above inequality gives

$$\mathcal{E}[\psi_0] \geqslant \left(1 - \frac{1}{3}\tilde{C}|\rho|\right)\|\psi'\|^2 - |\gamma|\|\psi'\|.$$

Since in case (iii) $1 - \frac{1}{3}\tilde{C}|\rho| > 0$, hence $\|\psi'\| \leqslant C, \forall\, t \geqslant 0$. In order to consider case (iv), in which $\mu > 2$, we remark that (21) can be written as

$$\mathcal{E}[\psi_0] \geqslant z^2 - |\gamma|z - \tilde{C}\frac{|\rho|}{\mu+1}z^{\mu}, \qquad z = \|\psi\| \in \mathbb{R}^+, \quad \mu > 2.$$

Let $\mathcal{S}$ be the set of solutions of such inequality. If ρ is small enough then there exists $c_1, \tilde{C}$, $0 < c_1 < c_2$, such that $\|\psi_0{}'\| \leqslant c_1$ and $\mathcal{S} \cap [c_1, c_2] = \emptyset$. Hence, $\|\psi'\| \leqslant c_1, \forall\, t \geqslant 0$, proving (iv). Finally, in order to consider case (v), we set

$$\xi = \|\psi'\|^2, \qquad \eta = \|\psi\|_{2\mu+2}^{2\mu+2}, \qquad \zeta = |\psi(0)|^2$$

then the energy and the second derivative of the variance take the form

$$\mathcal{E}[\psi] = \xi + \frac{\rho}{\mu+1}\eta + \gamma\zeta. \tag{22}$$

$$\ddot{I} = 8\mathcal{E}[\psi] + 4\rho\frac{\mu-2}{\mu+1}\eta - 4\gamma\zeta. \tag{23}$$

If $\gamma > 0$, then at any time $\ddot{I} < 8\mathcal{E}[\psi_0] < 0$. Following (11), blow up is proven. Conversely, if $\gamma < 0$, then from equation (22) it follows that

$$\frac{\rho}{\mu+1}\eta = \mathcal{E}[\psi] - \xi - \gamma\zeta.$$

Plugging this equation into (23) and using inequality (A.1) we obtain

$$\begin{aligned}\ddot{I} &= 8\mathcal{E}[\psi] + 4\rho\frac{\mu-2}{\mu+1}\eta - 4\gamma\zeta \\ &= 8\mathcal{E}[\psi] + 4(\mu-2)[\mathcal{E}[\psi] - \xi - \gamma\zeta] - 4\gamma\zeta \\ &= -4(\mu-2)\xi + 4\mu\mathcal{E}[\psi] + 4(1-\mu)\gamma\zeta \\ &\leqslant -4(\mu-2)\xi + 4\mu\mathcal{E}[\psi] + 4(1-\mu)\gamma\sqrt{\xi}.\end{aligned}$$

Now, setting

$$a = 4(\mu-2) > 0, \qquad b = 4(1-\mu)\gamma > 0$$

and

$$c = -4\mu\mathcal{E}[\psi] > 0 \quad \text{since} \quad \mathcal{E}[\psi] < 0$$

then we obtain the inequality

$$\ddot{I} \leqslant -a\xi + b\sqrt{\xi} - c.$$

Hence, if a, b and c satisfy

$$b^2 < 4ac \tag{24}$$

then there exists $C > 0$ such that

$$\ddot{I} < -C, \quad \forall\, t \geqslant 0.$$

Condition (24) implies blow-up; that is, if $\mu > 2$, $\gamma \in \mathbb{R}$ and $\rho > 0$ are such that

$$\frac{(\mu-1)^2}{4\mu(\mu-2)} < -\frac{\mathcal{E}[\psi]}{\gamma^2} \tag{25}$$

then we have blow-up in finite time. In particular, from (6) we have blow-up if $|\rho|$ is large enough. □

5. Concluding remarks

Remark 6. From theorem 3 it follows that no blow-up can occur for the physically most important cases $\mu = \frac{1}{2}$ and $\mu = 1$. Furthermore, we emphasize that, as for the free model (i.e., $\gamma = 0$), blow-up could occur only in the critical ($\mu = 2$) and hyper-critical ($\mu > 2$) cases and for attractive nonlinearity large enough.

Remark 7. In the case $\gamma < 0$, the results (iv) and (v) discussed in theorem 3 can be investigated in more detail when the initial data ψ_0 coincide with the ground state ϕ_γ for the Hamiltonian H_γ, namely

$$\psi_0(x) = \sqrt{\frac{|\gamma|}{2}}\,\mathrm{e}^{-|\gamma|\cdot|x|/2} \in \mathcal{K}$$

with associated eigenvalues

$$\mathcal{E}_{\mathrm{lin}}[\psi_0] = -\tfrac{1}{4}\gamma^2$$

An explicit computation gives

$$\mathcal{E}[\psi] = \mathcal{E}[\psi_0] = -\frac{1}{4}\gamma^2 - |\rho|\frac{|\gamma|^\mu}{2^\mu(\mu+1)^2} < 0$$

$$I(0) = \int_{\mathbb{R}} x^2(\psi_0(x))^2\,\mathrm{d}x = \frac{2}{\gamma^2}$$

$$\dot{I}(0) = 4\,\mathrm{Im}\int_{\mathbb{R}} x\bar{\psi}_0(x)\psi_0{}'(x)\,\mathrm{d}x = 0$$

$$\ddot{I}(0) = -4|\rho|\frac{\mu|\gamma|^\mu}{2^\mu(\mu+1)^2} < 0.$$

Then, inequality (25) implies that if the parameters $\mu > 2$ and $\rho < 0$ satisfy the following condition,

$$\frac{(\mu-1)^2}{4\mu(\mu-2)} < \frac{1}{4} + |\rho|\frac{|\gamma|^{\mu-2}}{2^\mu(\mu+1)^2}$$

then the solution ψ blows up in finite time.

Remark 8. By means of the same ideas it is possible to consider the occurrence of blow-up phenomena for the case with *nonlinear* pointwise interaction; that is, with Hamiltonian

$$H\psi = -\psi'' + V\psi, \qquad V_\nu\psi = \gamma|\psi|^{2\nu}\psi\delta_0 + \rho|\psi|^{2\mu}\psi, \qquad \nu, \quad \mu \geqslant 0, \quad \gamma, \quad \rho \in \mathbb{R}.$$

If we assume existence and uniqueness for the solution ψ_t to the associated Cauchy problem, then the conservation laws of L^2-norm and energy (12) imply that there is no blow-up provided that one the following conditions is satisfied:

- $\gamma, \rho \geqslant 0$;
- $\gamma \geqslant 0$ and $\rho < 0$ and $\mu < 2$;
- $\rho \geqslant 0$ and $\gamma < 0$ and $\nu < 1$;
- $\gamma, \rho < 0$ and $\nu < 1$ and $\mu < 2$.

Indeed, in such cases inequality (20) takes the form

$$\mathcal{E}[\psi] \geqslant \|\psi'\|^2 - \gamma_-\|\psi'\|^{\nu+1} - \tilde{C}\rho_-\|\psi'\|^\mu$$

where

$$\gamma_- = \max(0, -\gamma), \qquad \rho_- = \max(0, -\rho),$$

and an *a priori* estimate for the norm $\|\psi'\|$ follows. In contrast, the same arguments of theorem 2 give that the formal derivation of the second derivative of the variance reads

$$\ddot{I} = 8\mathcal{E}[\psi] + 4\rho\frac{\mu-2}{\mu+1}\|\psi\|_{2\mu+2}^{2\mu+2} + 4\gamma\frac{\nu-1}{\nu+1}|\psi(0)|^{2\nu+2}.$$

Hence, blow-up occurs when $\mathcal{E}[\psi] < 0$, $\mu > 2$, and $\nu > 1$.

Acknowledgments

This work is partially supported by the INdAM-GNFM project *Comportamenti Classici in Sistemi Quantistici*. RA is supported by a Marie Curie Reintegration Grant, contract no ERG-508032.

Appendix. Inequalities

We recall the following inequalities. Let $\psi \in L^2(\mathbb{R}) \cap H^1(\mathbb{R})$, then

$$\|\psi\|_\infty^2 \leqslant \|\psi\| \cdot \|\psi'\|. \tag{A.1}$$

If $\psi \in L^2(\mathbb{R}^d) \cap H^1(\mathbb{R}^d)$,

$$\|\psi\|_{2\chi+2}^{2\chi+2} \leqslant \tilde{C}\|\nabla\psi\|^{\chi d}\|\psi\|^{2+\chi(2-d)} \tag{A.2}$$

for some positive constant $\tilde{C} = \tilde{C}(\chi, d)$, where

$$\chi \in \begin{cases} [0, +\infty] & \text{if} \quad d = 1 \\ [0, +\infty) & \text{if} \quad d = 2 \\ [0, 2/(d-2)) & \text{if} \quad d > 2 \end{cases}$$

and $\tilde{C} = 4\pi^{-2}$ for $d = 1$.

References

[1] Adami R and Teta A 2001 A class of nonlinear Schrödinger equations with concentrated nonlinearity *J. Funct. Anal.* **180** 148–75

[2] Adami R, Dell'Antonio G, Figari R and Teta A 2003 The Cauchy problem for Schrödinger equation in dimension three with concentrated nonlinearity *Ann. Inst. H Poincare* **20** 477–500

[3] Adami R, Dell'Antonio G, Figari R and Teta A 2004 Blow up solutions for Schrödinger equation in dimension three with a concentrated nonlinearity *Ann. Inst. H Poincare* **21** 121–37

[4] Albeverio S, Brzeźniak Z and Dąbrowski L 1994 Time-dependent propagator with point interaction *J. Phys. A: Math. Gen.* **27** 4933–43

[5] Albeverio S, Gesztesy F, Hoegh-Krohn R and Holden H 1988 *Solvable Models in Quantum Mechanics* (Berlin: Springer)

[6] Carles R 2002 Remarks on nonlinear Schrödinger equations with harmonic potentials *Ann. Inst. H Poincare* **3** 757–72

[7] Carles R 2002 Critical nonlinear Schrödinger equations with and without harmonic potentials *Math. Models Methods Appl. Sci.* **12** 1513–23

[8] Cazenave T 2003 *Semilinear Schrödinger Equations (Courant Lecture Notes in Mathematics)* (New York: Courant Institute of Mathematical Sciences)

[9] Dodd R K, Eilbeck J C, Gibbon J D and Morris H C 1982 *Solitons and Nonlinear Wave Equations* (London: Academic)

[10] Fermi E 1936 Sul moto dei neutroni nelle sostanze idrogenate *Ric. Sci.* **7** 13–52 (in Italian)

[11] Glassey R T 1977 On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations *J. Math. Phys.* **18** 1794–7

[12] Leboeuf P and Pavloff N 2001 Bose–Einstein beams: coherent propagation through a guide *Phys. Rev.* A **64** 033602

[13] Lieb E H, Seiringer R and Yngvason J 2004 One-dimensional behavior of dilute, trapped Bose gases *Commun. Math. Phys.* **244** 347–93

[14] Merle F and Raphael P 2005 Profiles and quantization of the blow up mass for critical nonlinear Schrödinger equation *Commun. Math. Phys.* **253** 675–704

Merle F and Raphael P 2004 On universality of blow-up profile for L^2 critical nonlinear Schrödinger equation *Invent. Math.* **156** 565–672

Merle F and Raphael P 2003 Sharp upper bound on the blow-up rate for the critical nonlinear Schrödinger equation *Geom. Funct. Anal.* **13** 591–642

[15] Pitaevskii L P and Stringari S 2003 *Bose–Einstein Condensation* (Oxford: Oxford University Press)

[16] Reed M and Simon B 1975 *Methods of Modern Mathematical Physics. II. Fourier Analysis, Self-adjointness* (New York: Academic)

[17] Schulman L S 1986 Application of the propagator for the delta function potential *Path Integrals from meV to MeV* ed M C Gutzwiller, A Ioumata, J K Klauder and L Streit (Singapore: World Scientific) pp 302–11

[18] Sulem C and Sulem P L 1999 *The Nonlinear Schrödinger Equation: Self-focusing and Wave Collapse* (New York: Springer)

[19] Witthaut D, Mossmann S and Korsch H J 2005 Bound and resonance states of the nonlinear Schrödinger equation in simple model systems *J. Phys. A: Math. Gen.* **38** 1777–92